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ELSEVIER
Linear Algebra and its Applications 430 (2009) 423–437
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Tools for the eigenvalue distribution in a non-Hermitian setting

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Received 15 January 2008; accepted 8 August 2008
Available online 24 September 2008
Submitted by R.A. Brualdi

Abstract

Under mild trace norm assumptions on the perturbing sequence, we extend a recent perturbation result based on a theorem by Mirsky. The analysis concerns the eigenvalue distribution and localization of a generic (non-Hermitian) complex perturbation of a bounded Hermitian sequence of matrices. Some examples of application are considered, ranging from the product of Toeplitz sequences to the approximation of PDEs with given boundary conditions. A final discussion on open questions and further lines of research ends the note.

AMS classification: 15A18; 15A12; 47B36; 47B65

Keywords: Matrix sequence; Joint eigenvalue distribution; Mirsky theorem; Toeplitz matrix; Approximation of PDEs

1. Introduction

In recent works, the notion of approximating class of sequences was introduced as reported below.

Definition 1.1 [10]. Suppose a sequence of matrices $\{A_n\}$, A_n of size d_n, is given, $d_k < d_{k+1}$ for each k. We say that $\{\{B_{n,m}\} : m \in \mathbb{N}^+\}$, $B_{n,m}$ of size d_n, is an approximating class of sequences

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[1] The work of the authors is supported in part by MIUR Grant No. 2006017542.

0024-3795/$ - see front matter
doi:10.1016/j.laa.2008.08.006

($a.c.s.$) for $\{A_n\}$ if, for all sufficiently large $m \in \mathbb{N}^+$, the following splitting holds:

$$A_n = B_{n,m} + R_{n,m} + N_{n,m} \quad \text{for all } n > n_m,$$

with

$$\text{rank } R_{n,m} \leqslant d_n\, c(m), \quad \|N_{n,m}\| \leqslant \omega(m),$$

where $\|\cdot\|$ is the spectral norm (maximal singular value), n_m, $c(m)$ and $\omega(m)$ depend only on m and, moreover,

$$\lim_{m\to\infty} \omega(m) = 0, \quad \lim_{m\to\infty} c(m) = 0.$$

The idea behind the concept of $a.c.s.$ was to define a basic approximation theory for matrix sequences with respect to the global distribution of eigenvalues and singular values. More precisely, given a "difficult" sequence $\{A_n\}$, the goal is to recover its global spectral behavior from the spectral behavior of simpler approximating sequences. Indeed, in accordance with our notion of approximation described in Definition 1.1, the following approximation result holds.

Proposition 1.2 [10]. *Let $\{d_n\}$ be an increasing sequence of natural numbers. Suppose a sequence of Hermitian matrices $\{A_n\}$ is given such that A_n is of size d_n and $\{\{B_{n,m}\} : m \in \mathbb{N}^+\}$ is an $a.c.s.$ for $\{A_n\}$ in the sense of Definition 1.1, with all $B_{n,m}$ being Hermitian. Suppose that $\{B_{n,m}\} \sim_\lambda (h_m, K)$ and that h_m converges in measure to the measurable function h over K, K of finite and positive measure. Then necessarily*

$$\{A_n\} \sim_\lambda (h, K).$$

Here we say that a sequence $\{A_n\}$ is distributed as a measurable function h over its domain K with positive and finite (Lebesgue) measure $\mu\{K\}$, and we write $\{A_n\} \sim_\lambda (h, K)$ if and only if for every $F \in \mathscr{C}_0(\mathbb{C})$ (continuous with bounded support over the complex field $\mathbb{C}$)

$$\lim_{n\to\infty} \Sigma_\lambda(F, A_n) = \frac{1}{\mu\{K\}} \int_K F(h(s))\,\mathrm{d}s,$$

where

$$\Sigma_\lambda(F, A_n) = \frac{1}{d_n} \sum_{i=1}^{d_n} F(\lambda_i(A_n)),$$

with $\{\lambda_i(A_n)\}$ denoting the set of the eigenvalues of A_n (counted with their multiplicities) and with the integral performed with respect to the same measure $\mu\{\cdot\}$.

Now if we lose the Hermitian character either of A_n or of $B_{n,m}$, then the same statements as in Proposition 1.2 are true in full generality for the singular values (see [10,13] for details, more results, and applications), but become false in general when considering the eigenvalues; see [14] for a striking counterexample.

Our goal is to give new more restrictive conditions under which a more severe notion of approximating class of sequences still enables to derive the spectral distribution of a "difficult" sequence from those of simpler approximating sequences.

More precisely, under mild trace norm assumptions on the perturbing sequence, we extend a recent perturbation result based on a theorem by Mirsky. The analysis regards the localization and the distribution of the eigenvalues of a generic (non-Hermitian) complex perturbation of a bounded Hermitian sequence of matrices. Some examples of application are also considered, ranging from the sequence of component-wise products of Toeplitz sequences to the approximation of PDEs with various boundary conditions.

The paper is organized as follows: the main theoretical results are proven in Section 2. Sections 3 and 4 are devoted to applications. Section 5 contains a conclusive discussion.

2. Main results

We start with a few notions and a well-known inequality.

Definition 2.1. A matrix sequence $\{A_n\}$ of size d_n, $d_k < d_{k+1}$ for each k, is *properly* (*or strongly*) *clustered at* $s \in \mathbb{C}$ (in the eigenvalue sense), if for any $\epsilon > 0$ the number of the eigenvalues of A_n off the disk

$$B(s,\epsilon) := \{z : |z - s| < \epsilon\},$$

can be bounded by a pure constant q_ϵ possibly depending on ϵ, but not on n. In other words

$$q_\epsilon(n,s) := \#\{i : \lambda_i(A_n) \notin B(s,\epsilon)\} = \mathrm{O}(1), \quad n \to \infty.$$

If every A_n has only real eigenvalues (at least for all n large enough), then s is real and the disk $B(s,\epsilon)$ reduces to the interval $(s-\epsilon, s+\epsilon)$. Furthermore, $\{A_n\}$ is *properly* (*or strongly*) *clustered at a nonempty closed set* $S \subset \mathbb{C}$ (in the eigenvalue sense) if for any $\epsilon > 0$

$$q_\epsilon(n,S) := \#\{i : \lambda_i(A_n) \notin B(S,\epsilon)\} = \mathrm{O}(1), \quad n \to \infty,$$

$B(S,\epsilon) := \cup_{s\in S} B(s,\epsilon)$ is the ϵ-neighborhood of S, and if every A_n has only real eigenvalues, then S has to be a nonempty closed subset of $\mathbb{R}$. Finally, the term “properly (or strongly)” is replaced by “weakly”, if

$$q_\epsilon(n,s) = o(d_n), \qquad (q_\epsilon(n,S) = o(d_n)), \quad n \to \infty,$$

in the case of a point s (a closed set S), respectively.

To link the concept of cluster with the distribution notion it is instructive to observe that $\{A_n\} \sim_\lambda (h, K)$, with $h \equiv s$ being a constant function, is equivalent to write that $\{A_n\}$ is weakly clustered at $s \in \mathbb{C}$. Moreover in the following the definition of essential range will be used.

Definition 2.2. Given a measurable complex-valued function h defined on a Lebesgue measurable set K, the *essential range of* h is the set $\mathscr{S}(h)$ of points $s \in \mathbb{C}$ such that, for every $\varepsilon > 0$, the Lebesgue measure of the set $h^{(-1)}(B(s,\varepsilon)) := \{t \in K : h(t) \in B(s,\varepsilon)\}$ is positive. The function h is *essentially bounded* if its essential range is bounded.

$\mathscr{S}(h)$ is clearly a closed set (its complement is open), and moreover

$$\mathscr{S}(h) = \bigcap\{D : \text{closed set with } \mu\{h^{(-1)}(D)\} = \mu\{K\}\}.$$

We need also the well-known inequality

$$|\mathrm{tr}(A)| \leqslant \|A\|_1, \tag{1}$$

where A is any square matrix of size n, $\mathrm{tr}(A)$ is the trace of A, i.e., the sum of all its diagonal entries (or equivalently the sum of all its eigenvalues), and $\|A\|_1$ is its trace norm, i.e., the sum of all its singular values.

A simple proof of (1) is as follows. Let $A = USV$ be the singular value decomposition of A [4]. Then by a similarity argument

$$\operatorname{tr}(A) = \operatorname{tr}(USV) = \operatorname{tr}(SVU) = \operatorname{tr}(SW),$$

with $W = VU$ being unitary. So, $|\operatorname{tr}(A)| = |s_1 w_1 + s_2 w_2 + \cdots + s_n w_n|$, with $s_1, s_2, \ldots, s_n$ being the singular values of A and where $w_1, w_2, \ldots, w_n$ are the diagonal entries of W: all of them bounded by 1. Hence, the application of the triangle inequality yields (1).

Furthermore, given a square complex matrix A, we may always write

$$A = \operatorname{Re}(A) + \mathrm{i}\operatorname{Im}(A), \quad \mathrm{i}^2 = -1, \tag{2}$$

$$\operatorname{Re}(A) = (A + A^*)/2, \tag{3}$$

$$\operatorname{Im}(A) = (A - A^*)/(2i), \tag{4}$$

where as usual, A^* is the conjugate transpose of the matrix A. Here we notice that both $\operatorname{Re}(A)$ and $\operatorname{Im}(A)$ are Hermitian matrices by construction.

We are now ready for proving the main result. It is worthwhile to stress that such a statement could be seen as a generalization of Theorem 3.4, p. 93 in [7] when the notion of $a.c.s.$ is taken into consideration.

Theorem 2.3. *Let $\{\{B_{n,m}\}, m \in \mathbb{N}^+\}$ be an $a.c.s.$ for $\{A_n\}$(A_n, $B_{n,m}$ of size d_n) such that $E_{n,m} = N_{n,m} + R_{n,m}$, $B_{n,m}$ are Hermitian and*

$$\{B_{n,m}\} \sim_\lambda (h_m, K), \quad 0 < \mu\{K\} < \infty,$$

$$\lim_{m\to\infty} h_m = h \text{ in measure on } K, \tag{5}$$

with

$$\sup_m \sup_n \|B_{n,m}\| = \widetilde{C},$$

$$\sup_m \sup_n \|E_{n,m}\| = \widehat{C},$$

$$C = \max\{\widetilde{C}, \widehat{C}\}.$$

Here $\widetilde{C}$, $\widehat{C}$ are positive universal constants, $\|E_{n,m}\|_1 \leqslant c(m)d_n$ with $c(m) \underset{m\to\infty}{\to} 0$ ($\|\cdot\|_1$ being the trace norm, i.e., Schatten 1 norm).

Then h is real-valued and $\{A_n\}$ is distributed as h, i.e., $\{A_n\} \sim_\lambda (h, K)$ or, equivalently,

$$\lim_{n\to\infty} \Sigma_\lambda(F, A_n) = \frac{1}{\mu\{K\}} \int_K F(h(x))\,\mathrm{d}x \tag{6}$$

$\forall F \in \mathscr{C}_0(\mathbb{C})$.

Proof. We define the functionals acting on $\mathscr{C}_0(\mathbb{C})$ as follows:

$$\Phi_m(F) = \frac{1}{\mu\{K\}} \int_K F(h_m(x))\,\mathrm{d}x,$$

$$\Phi(F) = \frac{1}{\mu\{K\}} \int_K F(h(x))\,\mathrm{d}x,$$

where the function F is continuous with bounded support, i.e., $F \in \mathscr{C}_0(\mathbb{C})$. It is immediate to check that relation (6) is equivalent to write that $\forall \epsilon > 0$, $\exists \bar{n} > 0$ such that $\forall n \geqslant \bar{n}$ and $\forall F \in \mathscr{C}_0(\mathbb{C})$ we have $|\Sigma_\lambda(F, A_n) - \Phi(F)| < \epsilon$. By allowing the parameter m, the latter is equivalent to state that $\exists k(m) \underset{m\to\infty}{\to} 0$ such that $\forall F \in \mathscr{C}_0(\mathbb{C})$, $\forall m \in \mathbb{N}^+$ there exists $\bar{n}_m \in \mathbb{N}$ and the inequalities

$$|\Sigma_\lambda(F, A_n) - \Phi(F)| < k(m) \quad \forall n \geqslant \bar{n}_m, \tag{7}$$

are fulfilled. Let us consider the left-hand side of (7) and let us decompose it in basic quantities to be studied separately. In fact, by proper manipulations we find

$$\begin{aligned}|\Sigma_\lambda(F, A_n) - \Phi(F)| &= |\Sigma_\lambda(F, A_n) - \Sigma_\lambda(F, B_{n,m}) + \\ &\quad +\Sigma_\lambda(F, B_{n,m}) - \Phi_m(F) + \\ &\quad +\Phi_m(F) - \Phi(F)| \\ &\leqslant \alpha_{n,m} + \beta_{n,m} + \gamma_m,\end{aligned}$$

where

$$\begin{aligned}\alpha_{n,m} &= |\Sigma_\lambda(F, A_n) - \Sigma_\lambda(F, B_{n,m})|, \\ \beta_{n,m} &= |\Sigma_\lambda(F, B_{n,m}) - \Phi_m(F)|, \\ \gamma_m &= |\Phi_m(F) - \Phi(F)|.\end{aligned}$$

First let us focus on the quantities $\beta_{n,m}$. From the assumptions in (5) for all fixed m, $\beta_{n,m}$ converges to zero as $n \to \infty$; then we can take a value $\hat{n}_m$ sufficiently large in such a way that $\beta_{n,m} \leqslant \frac{1}{m}$ $\forall n \geqslant \hat{n}_m$. In other words we can write $\limsup_{n\to\infty} \beta_{n,m} \leqslant \frac{1}{m}$. By following the same reasoning on the quantities γ_m, by (5) and from [17] (Remark 5.1.3, p. 124), $\lim_{m\to\infty} \Phi_m(F) = \Phi(F)$, we can write $\gamma_m = |\Phi_m(F) - \Phi(F)| \leqslant \upsilon(m)$, with $\upsilon(m) \to 0$ as $m \to \infty$.

As a consequence, the proof of (7) (and a fortiori of (6)) is reduced to check whether $[\limsup_{n\to\infty} \alpha_{n,m}] \underset{m\to\infty}{\to} 0$, that is

$$|\Sigma_\lambda(F, A_n) - \Sigma_\lambda(F, B_{n,m})| \leqslant \delta(m), \quad \delta(m) \underset{m\to\infty}{\to} 0. \tag{8}$$

In conclusion the proof of (8) will consist in verifying the assumptions of Theorem 2.2, p. 88 in [7]. Hence, we perform the following steps.

I. The spectrum of all A_n is uniformly bounded, that is $\exists Q$ positive constant such that $|\lambda_j(A_n)| < Q$ $\forall n$ ($\lambda_j \in \Sigma(A_n)$, where $\Sigma(A_n)$ denotes the set of all the eigenvalues of A_n);
II. Relation (6) is verified whenever F is a polynomial of arbitrary fixed degree;
III. The sequence $\{A_n\}$ is weakly clustered, in the eigenvalue sense, at a compact set $S \subset \mathbb{C}$ with empty interior, such that $\mathbb{C}\backslash S$ is a connected set, and $\mathscr{S}(h) \subset S$, with $\mathscr{S}(h)$ denoting the essential range of h.

Item I. From the assumptions we have

$$\|A_n\| = \|B_{n,m} + E_{n,m}\| \leqslant \|B_{n,m}\| + \|E_{n,m}\| \leqslant 2C \quad \forall n, m,$$

and hence the spectra of the sequences $\{A_n\}$, $\{B_{n,m}\}$, and $\{E_{n,m}\}$ lie all in the closed disk $\{|z| \leqslant 2C\}$. In particular, the spectrum of all A_n is uniformly bounded since $\forall n$ $|\lambda_j(A_n)| \leqslant 2C$, $2C$ constant independent of n, $\lambda_j \in \Sigma(A_n)$.

Item II. Since

$$\mathrm{tr}(X) = \sum_{\lambda \in \Sigma(X)} \lambda = \sum_{k=1}^{d_n} [X]_{k,k},$$

and since $\mathrm{tr}(\cdot)$ is a linear functional, the assumption $A_n = B_{n,m} + E_{n,m}$ implies that $\mathrm{tr}(A_n) - \mathrm{tr}(B_{n,m}) = \mathrm{tr}(E_{n,m})$. Consequently

$$\begin{aligned}\left|\frac{1}{d_n}\sum_{\lambda \in \Sigma(A_n)} \lambda - \frac{1}{d_n}\sum_{\lambda \in \Sigma(B_{n,m})} \lambda\right| &= \left|\frac{1}{d_n}\sum_{\lambda \in \Sigma(E_{n,m})} \lambda\right| \\ &\leqslant_{(\alpha)} \frac{1}{d_n}\|E_{n,m}\|_1 \\ &\leqslant_{(\beta)} \frac{1}{d_n} c(m) d_n = c(m) \underset{m\to\infty}{\to} 0,\end{aligned}$$

where (α) follows from (1) and (β) follows from the assumptions; therefore, by invoking also (8), we deduce that (6) is satisfied in the special case where $F(z) = z$ (defined on the whole complex field $\mathbb{C}$, hence with non-compact support, but which can be considered an admissible test function since the spectra of all A_n are uniformly bounded).

We now prove that (6) is satisfied by taking as test function F any arbitrary polynomial of fixed degree. To this end, from the linearity in the first variable of the operators $\Sigma_\lambda(\cdot,\cdot)$ and $\Phi(\cdot)$ (i.e. $\Sigma_\lambda(aG + bH, \cdot) = a\Sigma_\lambda(G, \cdot) + b\Sigma_\lambda(H, \cdot)$ and $\Phi(aG + bH) = a\Phi(G) + b\Phi(H)$), it is sufficient to consider the case of monomials, i.e., $F(z) = z^q$ for all non-negative integers q. For $q = 0, 1$ the result is valid, so that we focus our attention to the case where $q \geqslant 2$. Relation $A_n = B_{n,m} + E_{n,m}$ implies

$$A_n^q = (B_{n,m} + E_{n,m})^q = B_{n,m}^q + \widetilde{E}_{n,m},$$

where $\widetilde{E}_{n,m}$ is a term of the form

$$\widetilde{E}_{n,m} = \sum_{X_i \in \{B_{n,m}, E_{n,m}\}} (X_1 \cdots X_q) - B_{n,m}^q. \tag{9}$$

In other words, the error matrix $\widetilde{E}_{n,m}$ is the sum of all possible combinations of products of j matrices $B_{n,m}$ and k matrices $E_{n,m}$, with $j + k = q$ and the exception of $j = q$ (obviously it is understood that all the addends are pairwise different). By using a simple Hölder inequality involving Schatten p norms: $\|XY\|_1 \leqslant \|X\|\|Y\|_1$, for every summand R in (9), we deduce that there exists $j \geqslant 1,\ k = q - j$ for which

$$\begin{aligned}\|R\|_1 &\leqslant \|B_{n,m}\|^k \|E_{n,m}\|^{j-1} \|E_{n,m}\|_1 \\ &\leqslant C^k C^{j-1} c(m) d_n.\end{aligned} \tag{10}$$

Therefore, by the triangle inequality and by applying inequality (10) to any summand in (9), we find $\|\widetilde{E}_{n,m}\|_1 \leqslant \widehat{K} c(m) d_n$, with $\widehat{K} = \widehat{K}(q)$ constant independent of n and m. Consequently $\mathrm{tr}(A_n^q) - \mathrm{tr}(B_{n,m}^q) = \mathrm{tr}(\widetilde{E}_{n,m})$, and, since $\lambda(X^q) = \lambda^q(X)$, we have

$$\begin{aligned}\left|\frac{1}{d_n}\sum_{\lambda\in\Sigma(A_n)}\lambda^q-\frac{1}{d_n}\sum_{\lambda\in\Sigma(B_{n,m})}\lambda^q\right| &= \left|\frac{1}{d_n}\sum_{\lambda\in\Sigma(\widetilde{E}_{n,m})}\lambda\right|\\ &\leqslant \frac{1}{d_n}\|\widetilde{E}_{n,m}\|_1\\ &\leqslant \frac{1}{d_n}\widehat{K}c(m)d_n=\widehat{K}c(m)\underset{m\to\infty}{\to}0.\end{aligned}$$

The latter joint with relation (8) proves that (6) is satisfied with $F(z)=z^q$ for any non-negative integer q and, a fortiori, with any polynomial F of fixed degree.

Item III. According to the standard notations in (2)–(4), we write the matrix $E_{n,m}$ as

$$E_{n,m}=\mathrm{Re}(E_{n,m})+\mathrm{i}\,\mathrm{Im}(E_{n,m}).$$

Clearly we have

$$\|\mathrm{Re}(E_{n,m})\|_1\leqslant\|E_{n,m}\|_1\leqslant c(m)d_n, \tag{11}$$

$$\|\mathrm{Im}(E_{n,m})\|_1\leqslant\|E_{n,m}\|_1\leqslant c(m)d_n. \tag{12}$$

We recall that every matrix $B_{n,m}$ is Hermitian and the same is obviously true for $\mathrm{Re}(A_n)$ and $\mathrm{Re}(E_{n,m})$. Since

$$\|\mathrm{Re}(A_n)-B_{n,m}\|_1=\|\mathrm{Re}(E_{n,m})\|_1\leqslant_{(11)}c(m)d_n,$$

from Lemma 5.1.3, p. 126, and from Corollary 5.1.2, p. 128 in [17], we deduce that $\{\{B_{n,m}\}, m\in\mathbb{N}^+\}$ is an $a.c.s.$ for the sequence $\{\mathrm{Re}(A_n)\}$. From (5) and from Corollary 5.1.1, p. 124 in [17], it follows that

$$\{\mathrm{Re}(A_n)\}\sim_\lambda(h,K).$$

As a consequence from Theorem 2.4, p. 90 in [7], $\{\mathrm{Re}(A_n)\}$ is weakly clustered at the essential range $\mathscr{S}(h)$ of h, which is a compact subset of $[-2C,2C]$ (recall that $\max|\lambda(\mathrm{Re}(A_n))|=\|\mathrm{Re}(A_n)\|\leqslant\|A_n\|\leqslant 2C$). Therefore all the eigenvalues of the Hermitian matrix $\mathrm{Re}(A_n)$ belong to the same interval $[-2C,2C]$.

We now consider the matrix $\mathrm{Im}(A_n)=\mathrm{Im}(E_{n,m})$. From (12) we have

$$\|\mathrm{Im}(A_n)\|_1=\|\mathrm{Im}(E_{n,m})\|_1\leqslant c(m)d_n,$$

from Lemma 5.1.3, p. 126 in [17], it follows that $\{0_n\}$ (sequence of null matrices of increasing size) is an $a.c.s.$ for the sequence $\{\mathrm{Im}(A_n)\}$, and since $\{0_n\}\sim_\lambda(0,K)$, from Corollary 5.1.1, p. 124 in [17], we deduce that

$$\{\mathrm{Im}(A_n)\}\sim_\lambda(0,K).$$

Therefore Theorem 2.4, p. 90 in [7], implies that $\{\mathrm{Im}(A_n)\}$ is weakly clustered a $\mathscr{S}(0)=\{0\}$. Therefore, by the definition of weak cluster, setting

$$B(s,\epsilon):=\{z:|z-s|<\epsilon\}\text{ and }B(S,\epsilon):=\cup_{s\in S}B(s,\epsilon)$$

for all $\epsilon>0$, we obtain that

$$\sharp\{j:\ \lambda_j(\mathrm{Im}(A_n))\notin B(0,\epsilon)\}=o(d_n). \tag{13}$$

Finally, from Ky Fan-Mirsky Theorem (see Theorem 3.1, p. 92 in [7]), and from (13), we plainly deduce

$$\sharp\{j : \operatorname{Im}(\lambda_j(A_n)) \notin B(0, \epsilon)\} = o(d_n).$$

Consequently from (10) and (11), pp. 92–93 in [7] and from Corollary 3.3, p. 93 in [7], we infer that all eigenvalues of A_n show real parts in the interval $[-2C, 2C]$ and

$$\sharp\{j : \lambda_j(A_n) \notin B([-2C, 2C], \epsilon)\} = o(d_n),$$

i.e., $\{A_n\}$ is weakly clustered at the compact set $[-2C, 2C]$, the interior of $[-2C, 2C]$ is empty as subset of $\mathbb{C}$, $\mathbb{C}\backslash[-2C, 2C]$ is connected in $\mathbb{C}$, and $\mathscr{S}(h) \subset [-2C, 2C]$.

Since all the assumptions of Theorem 2.2, p. 88 in [7] are met, we conclude that $\{A_n\}$ is distributed, in the eigenvalue sense, as the function h in its definition domain K. $\square$

3. The multilevel Toeplitz setting

Let f be a d variate complex-valued integrable function, defined over the hypercube Q^d, with $Q = (-\pi, \pi)$ and $d \geqslant 1$. From the Fourier coefficients of f

$$a_j = (2\pi)^{-d} \int_{Q^d} f(t)\mathrm{e}^{-\mathrm{i}\,(j,t)}\,\mathrm{d}t, \quad \mathrm{i}^2 = -1, \quad j = (j_1, \ldots, j_d) \in \mathbb{Z}^d,$$

with $(j, t) = \sum_{k=1}^{d} j_k t_k$, $n = (n_1, \ldots, n_d) \in \mathbb{N}_+^d$ and $N(n) = n_1 \cdots n_d$, we define the sequence of Toeplitz matrices $\{T_n(f)\}$, where $T_n(f) = [a_{j-r}]_{r,j=e^T}^{n}$ of size $N(n)$, $e^T = (1, \ldots, 1) \in \mathbb{N}^d$ is the Toeplitz matrix of order $N(n)$ generated by f (see [22]). We are interested in proving results regarding the asymptotic (as the multi-index n tends to infinity) spectral behavior of matrix sequences obtained by matrix operations (especially algebraic sum, multiplication) on Toeplitz sequences. When we write

$$n \to \infty,$$

with $n = (n_1, \ldots, n_d)$ being a multi-index, we mean that

$$\min_{1 \leqslant j \leqslant d} n_j \to \infty.$$

In the following theorem, we are interested in estimating $\|A_n - T_n(h)\|_1$, i.e., the Schatten 1 norm of $A_n - T_n(h)$, where $A_n = T_n(f)T_n(g)$, $n = (n_1, \ldots, n_d) \in \mathbb{N}_+^d$, $f, g \in L^\infty(Q^d)$, and $h = fg$. To this end we will essentially use only classical approximation results.

Theorem 3.1. *Let $f, g \in L^\infty(Q^d)$, $A_n = T_n(f)T_n(g)$, $n = (n_1, \ldots, n_d) \in \mathbb{N}_+^d$, and let $h = fg$. Then $\|A_n - T_n(h)\|_1 = o(N(n))$, $N(n) = n_1 \cdots n_d$.*

Proof. For a given $\theta \in L^1(Q^d)$, let $p_{m,\theta}$ its Cesaro sum of degree $m = (m_1, \ldots, m_d) \in \mathbb{N}^d$ (i.e., the arithmetic average of Fourier sums of order $q = (q_1, \ldots, q_d) \in \mathbb{N}^d$ with $0 \leqslant q_j \leqslant m_j$, $j = 1, \ldots, d$; see [5]): from standard trigonometric series theory we know that $p_{m,\theta}$ converges in L^1 norm to θ as every m_j tends to infinity, $j = 1, \ldots, d$, and we also know that $\|p_{m,\theta}\|_{L^\infty} \leqslant \|\theta\|_{L^\infty}$, whenever $\theta \in L^\infty(Q^d) \subset L^1(Q^d)$. Furthermore, the norm inequality $\|T_n(\theta)\|_1 \leqslant N(n)(2\pi)^{-d}$ $\|\theta\|_{L^1}$ holds for every $\theta \in L^1(Q^d)$ and $\|T_n(\theta)\| \leqslant \|\theta\|_{L^\infty}$ holds for every $\theta \in L^\infty(Q^d)$ (see [16]). Hence, by adding and subtracting and by using repeatedly the triangle inequality

$$\begin{aligned}\|A_n - T_n(h)\|_1 \leqslant{} & \|A_n - T_n(p_{m,f})T_n(g)\|_1 + \|T_n(p_{m,f})T_n(g) - T_n(p_{m,f})T_n(p_{m,g})\|_1 \\ & + \|T_n(p_{m,f})T_n(p_{m,g}) - T_n(p_{m,f}\,p_{m,g})\|_1 + \|T_n(p_{m,f}\,p_{m,g}) - T_n(h)\|_1.\end{aligned}$$

Now, by using Hölder inequalities involving Schatten p norms: $\|XY\|_1 \leqslant \|X\|\|Y\|_1$, and the previous norm inequality from [16], we infer

$$\begin{aligned}\|A_n - T_n(p_{m,f})T_n(g)\|_1 &= \|(T_n(f) - T_n(p_{m,f}))T_n(g)\|_1 \\ &\leqslant \|T_n(f) - T_n(p_{m,f})\|_1 \|T_n(g)\| \\ &\leqslant N(n)(2\pi)^{-d} \|f - p_{m,f}\|_{L^1} \|g\|_{L^\infty},\end{aligned}$$

$$\begin{aligned}\|T_n(p_{m,f})T_n(g) - T_n(p_{m,f})T_n(p_{m,g})\|_1 &= \|T_n(p_{m,f})(T_n(g) - T_n(p_{m,g}))\|_1 \\ &\leqslant \|T_n(g) - T_n(p_{m,g})\|_1 \|T_n(p_{m,f})\| \\ &\leqslant \|T_n(g - p_{m,g})\|_1 \|p_{m,f}\|_{L^\infty} \\ &\leqslant N(n)(2\pi)^{-d} \|g - p_{m,g}\|_{L^1} \|f\|_{L^\infty},\end{aligned}$$

$$\begin{aligned}\|T_n(p_{m,f}p_{m,g}) - T_n(h)\|_1 &= \|T_n(h - p_{m,f}p_{m,g})\|_1 \\ &\leqslant N(n)(2\pi)^{-d} \|h - p_{m,f}p_{m,g}\|_{L^1}.\end{aligned}$$

The case of $\|T_n(p_{m,f})T_n(p_{m,g}) - T_n(p_{m,f}p_{m,g})\|_1$ can be treated separately. Indeed since all the involved symbols are trigonometric polynomials of degree not exceeding m, a direct check shows that the two matrices $T_n(p_{m,f})T_n(p_{m,g})$ and $T_n(p_{m,f}p_{m,g})$ can differ only on the first m_1 block rows and on the last m_1 block rows of size $N(n)/n_1$; moreover on every block of size $N(n)/n_1$ the two matrices can differ only the first m_2 block rows and on the last m_2 block rows of size $N(n)/(n_1n_2)$ and so on. Therefore, setting $\|m\|_\infty = \max_{1\leqslant j\leqslant d} m_j$, the trace norm of $T_n(p_{m,f})T_n(p_{m,g}) - T_n(p_{m,f}\ p_{m,g})$ is bounded by the rank times the spectral norm, i.e.,

$$\begin{aligned}&\|T_n(p_{m,f})T_n(p_{m,g}) - T_n(p_{m,f}p_{m,g})\|_1 \\ &\quad \leqslant 4\|m\|_\infty \|g\|_{L^\infty} \|f\|_{L^\infty} N(n)/(\min_j n_j) = o(N(n)).\end{aligned}$$

Finally, since the Cesaro operator converges to the identity in the L^1 topology, it follows that $\|f - p_{m,f}\|_{L^1}$ and $\|g - p_{m,g}\|_{L^1}$ can be made arbitrarily small as every m_j tends to infinity, $j = 1, \ldots, d$. Hence, for every $\epsilon > 0$, there exists $\bar{m}$ such that for any m larger than $\bar{m}$ in the component-wise sense, we have $\|A_n - T_n(h)\|_1 \leqslant \epsilon N(n)$, i.e., $\|A_n - T_n(h)\|_1 = o(N(n))$ and the proof is concluded. □

3.1. *The Toeplitz product $\{T_n(f)T_n(g)\}$ with bounded real-valued cumulative symbol $h = fg$*

Here we are interested in the eigenvalue distribution of component-wise products of Toeplitz sequences with real-valued symbols, whose study arises, e.g. in statistics (see [3,2]). Indeed, the main result follows from combining Theorems 3.1 and 2.3.

Theorem 3.2. *Let $f, g \in L^\infty(Q^d)$ be such that $h = fg$ is real-valued, $d \geqslant 1$. Then $\{A_n\} \sim_\lambda (h, Q^d)$ with $A_n = T_n(f)T_n(g)$.*

Proof. By [8] we know that $\{T_n(h)\} \sim_\lambda (h, Q^d)$ and $\|T_n(\theta)\| \leqslant \|\theta\|_{L^\infty}$ for every $\theta \in L^\infty(Q^d)$: therefore $\|T_n(h)\| \leqslant \|h\|_{L^\infty}$ and $\|A_n\| \leq \|T_n(f)\|\|T_n(g)\| \leq \|f\|_{L^\infty}\|g\|_{L^\infty}$. As a consequence of Theorem 3.1 we deduce that for all $m \geqslant 0$, $\|A_n - T_n(h)\|_1 \leqslant N(n)/m$ $\forall n \geqslant n_m$, where the relations $m \geqslant 0$ and $\forall n \geqslant n_m$ are intended component-wise. Therefore the desired results follow by applying Theorem 2.3 with $B_{n,m} = T_n(h)$, $E_{n,m} = A_n - T_n(h)$. $\square$

3.2. *The Toeplitz product $\{T_n(f)T_n(g)\}$ with cumulative symbol $h = fg$ in the Tilli class*

We start by introducing the Tilli class which is relevant for our subsequent spectral analysis.

Definition 3.3. Let D be any domain equipped with a positive measure and let us consider the space $L^\infty(D)$ of complex-valued essentially bounded functions. The *Tilli class* $\mathscr{T}$ is the subset of $L^\infty(D)$ made by functions whose (essential) range has empty interior and does not disconnect the complex plane.

It is clear that the condition defining the Tilli class does not involve any regularity of the function, but it is more related to the topology/geometry of the range (see also Example 5.39 in [6,24], at the top of page 390); by the way it is evident that the Tilli class includes properly all the real-valued L^∞ functions.

In the beautiful paper [21] Tilli was able to show that the distribution in the sense of the eigenvalues of the Toeplitz sequence $\{T_n(f)\}$ is valid whenever the symbol $f \in \mathscr{T}$. Indeed the proof is given in one dimension ($d = 1$) but the extension in several dimensions is plain.

Now we extend Theorem 3.2 from the subset of real-valued symbols to the whole Tilli class. The proof is not given in detail, but only by providing the main steps.

Theorem 3.4. *Let $f, g \in L^\infty(Q^d)$ be such that $h = fg$ belongs to the Tilli class, $d \geqslant 1$. Assume that a function ϕ can be found continuous on $\mathscr{S}(h)$, the range of h, such that it is injective and the range of $\phi(h)$ lies on the real line, i.e., $\mathscr{S}(\phi(h))$ is compact set of $\mathbb{R}$. Then $\{A_n\} \sim_\lambda (h, Q^d)$ with $A_n = T_n(f)T_n(g)$.*

Proof. We consider five steps:

Step 1. Given $\epsilon > 0$ consider ϕ_ϵ polynomial such that $\|\phi - \phi_\epsilon\|_{L^\infty,\mathscr{S}(h)} < \epsilon$, ϕ_ϵ is injective, the latter implying that its range does not disconnect the complex plane. In such a way the range of $\phi_\epsilon(h)$ lies in a ϵ-neighborhood of a compact subset of the real line.

Step 2. Therefore, $T_n(\phi(h))$ is Hermitian since $\phi(h)$ is real-valued and has real eigenvalues contained in the interval $[r, R]$ with r being the essential infimum of $\phi(h)$ and R being the essential supremum of $\phi(h)$. Moreover, since $\|T_n(\phi_\epsilon(h)) - T_n(\phi(h))\| = \|T_n(\phi_\epsilon(h) - \phi(h))\| \leqslant \|\phi(h) - \phi_\epsilon(h)\|_{L^\infty} = \|\phi - \phi_\epsilon\|_{L^\infty,\mathscr{S}(h)} < \epsilon$, it follows that the matrix $T_n(\phi_\epsilon(h))$ has all the eigenvalues in a ϵ-neighborhood of $[r, R]$.

Step 3. Now for every polynomial P of fixed degree $\|P(T_n(f)T_n(g)) - P(T_n(h))\|_1 = o(N(n))$ and $\|P(T_n(h)) - T_n(P(h))\|_1 = o(N(n))$; this is not difficult in view of Theorem 3.1.

Step 4. Using the previous step with any polynomial $P = \phi_\epsilon$ and since the eigenvalues of $T_n(\phi(h))$ belong to $[r, R]$, we deduce that the eigenvalues of $\{\phi_\epsilon(T_n(h))\}$ and $\{\phi_\epsilon(T_n(f)T_n(g))\}$ are clustered in a ϵ-neighborhood of $[r, R]$. As a consequence, the injectivity of ϕ_ϵ implies that the eigenvalues of $\{T_n(f)T_n(g)\}$ are clustered in a ϵ'-neighborhood of the range of $h = fg$. Since ϵ and therefore ϵ' can be chosen arbitrarily, it follows that the sequence $\{T_n(f)T_n(g)\}$ is clustered in the eigenvalue sense at $\mathscr{S}(h)$.

Step 5. By trivial computation it is easily deduced that for every positive integer k

$$\mathrm{tr}((T_n(f)T_n(g))^k - (T_n(h))^k) = o(N(n)).$$

Since $\{T_n(h)\} \sim_\lambda (h, Q^d)$ by the previous relation, for every positive integer k, it follows that

$$\lim_{n\to\infty} \frac{\operatorname{tr}(T_n(f)T_n(g))^k}{N(n)} = \frac{1}{(2\pi)^d}\int_{Q^d} h(s)^k\, \mathrm{d}s.$$

The above limit relation, the clustering of $\{T_n(f)T_n(g)\}$ at $\mathscr{S}(h)$, the uniform boundedness of the spectra of $\{T_n(f)T_n(g)\}$, and the fact that h does not disconnect the complex plane and its range has empty interior are the assumptions of Theorem 2.2 in [7]: the conclusion of Theorem 2.2 in [7] is exactly the desired claim and hence the proof is concluded. □

The proof would have worked without technical assumptions if the following claim would have been true.

Claim. *Given h bounded such that its range has empty interior and does not disconnect the complex plane, find ϕ continuous on $\mathscr{S}(h)$, the range of h, such that it is injective and the range of $\phi(h)$ lies on the real line, i.e., $\mathscr{S}(\phi(h))$ is compact set of $\mathbb{R}$.*

Unfortunately this claim is generally false as the subsequent example shows.

Proposition 3.5. *Let $K \subseteq \mathbb{C}$ be the Y shaped compact set illustrated in Fig. 1, and let $\phi : K \to \mathbb{R}$. Then ϕ continuous implies that ϕ is not injective and viceversa, i.e., ϕ injective implies that ϕ is not continuous.*

Proof. Let $\phi: K \to \mathbb{R}$ be continuous. Let us consider the three edges C_1, C_2, C_3 and the point α, $\alpha \notin C_r$, $r = 1, 2, 3$, as illustrated in Fig. 2, where $K = C_1 \cup C_2 \cup C_3 \cup \{\alpha\}$. The continuity of ϕ implies the following relationships

$$\lim_{\substack{z\in C_1\\ z\to\alpha}} \phi(z) = \phi(\alpha), \tag{14}$$

$$\lim_{\substack{z\in C_2\\ z\to\alpha}} \phi(z) = \phi(\alpha), \tag{15}$$

$$\lim_{\substack{z\in C_3\\ z\to\alpha}} \phi(z) = \phi(\alpha). \tag{16}$$

Assume now that ϕ is injective on K, that is $\forall z_1, z_2 \in K$, $z_1 \neq z_2$, we find $\phi(z_1) \neq \phi(z_2)$. Let us set $\phi(\alpha) = a$.

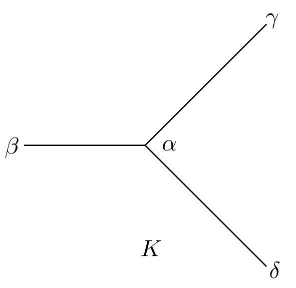

Fig. 1. Y shaped compact set *K*.

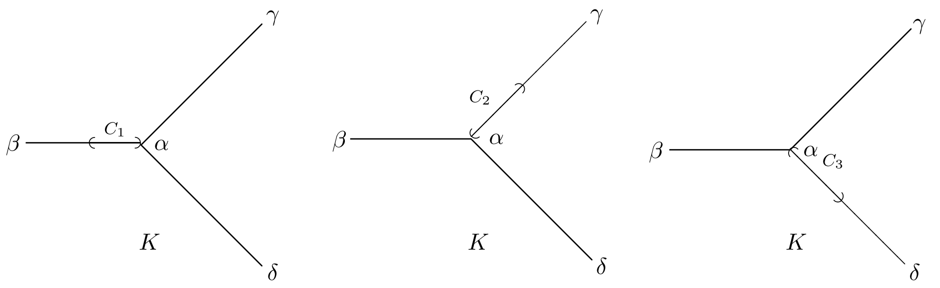

Fig. 2. Analysis of ϕ in the 3 branches of K.

The following reasonings can be made.

Claim 1. *Let us remark that, given any of the three sets C_r with $r = 1, 2, 3,$ the function $\phi - a$ cannot change sign since the existence of two points $z_1, z_2 \in C_r$ with $\phi(z_1) > a$ and $\phi(z_2) < a$, would imply by continuity the existence of $\tilde{z} \in C_r, \tilde{z} \neq \alpha (\alpha \notin C_r)$ such that $\phi(\tilde{z}) = \phi(\alpha) = a$. This would imply that ϕ is not injective.*

Claim 2. *Given C_r and $C_s, r, s = 1, 2, 3, r \neq s$, the following condition is impossible*:

$$\forall z \in C_r,\ \phi(z) > a \quad \text{and} \quad \forall t \in C_s,\ \phi(t) > a. \tag{17}$$

Indeed, suppose by contradiction that (17) is satisfied. Since relations (14), (15), and (16) hold, for every $\epsilon > 0$ there exist $\widetilde{C}_r \subseteq C_r$ and $\widehat{C}_s \subseteq C_s$, with $\widetilde{C}_r, \widehat{C}_s \neq \emptyset$, $\widetilde{C}_r \cap \widehat{C}_s = \emptyset$, such that

for $z \in \widetilde{C}_r$, the range of $\phi(z)$ contains $(a, a + \epsilon)$;

for $t \in \widehat{C}_s$, the range of $\phi(t)$ contains $(a, a + \epsilon)$.

Furthermore, since ϕ is a continuous function, by varying z in the set $\widetilde{C}_r$, $\phi(z)$ has to take all the values $(a, a + \epsilon)$ and by varying t in the set $\widehat{C}_s$, $\phi(t)$ has to take all the values $(a, a + \epsilon)$. From this it follows that there exist $\tilde{z} \in \widetilde{C}_r$ and $\hat{t} \in \widehat{C}_s$ with $\tilde{z} \neq \hat{t}$ such that $\phi(\tilde{z}) = \phi(\hat{t})$, and the latter would imply again that ϕ is not injective.

Claim 3. *Given C_r and $C_s, r, s = 1, 2, 3, r \neq s$, the following condition is impossible*:

$$\forall z \in C_r,\ \phi(z) < a \quad \text{and} \quad \forall t \in C_s,\ \phi(t) < a.$$

In fact, it is enough to repeat verbatim the same reasoning as in the previous claim, using the interval $(a - \epsilon, a)$ in place of $(a, a + \epsilon)$.

Let us consider for instance the subset C_1. Given relation (14) and Claim 1, we have two possibilities:

(I) $\forall z \in C_1, \phi(z) > a$;
(II) $\forall z \in C_1, \phi(z) < a$.

Let us suppose that case (I) holds: of course the alternative case (II) can be handled similarly. Therefore, we have

$$\forall z \in C_1, \quad \phi(z) > a. \tag{18}$$

Let us consider the subset C_2. With the same arguments used for C_1, from (15) and (18), Claim 2 implies that

$$\forall z \in C_2, \quad \phi(z) < a. \tag{19}$$

Now let us consider the subset C_3. From (16), given the continuity of ϕ, we deduce that

1. by Claim 1, $\phi - a$ cannot change sign in any of the subsets C_r, $r = 1, 2, 3$;
2. by Claim 2 and (18) $\forall z \in C_3, \phi(z) \not> a$;
3. by Claim 3 and (19) $\forall z \in C_3, \phi(z) \neq a$;
4. by injectivity $\forall z \in C_3, \phi(z) \neq a$.

The four listed claims are of course in contradiction. Therefore, ϕ cannot be simultaneously continuous and injective over all K and the proof is complete. □

However, even in this case where the key assumption of Theorem 3.4 is not satisfied, the Szegö formula can be recovered in full generality.

Indeed it is enough to repeat the same proof as in Theorem 3.4, with ϕ being continuous and injective over $C_1 \cup C_2$ and with $\phi(\alpha) = \phi(z)$ $\forall z \in C_3$. In that case we obtain a partial Szegö relation in which the test function is a arbitrary continuous function over $C_1 \cup C_2$, but it is constant over the remaining branch. However if we follow the same steps now choosing ϕ continuous and injective over $C_1 \cup C_3$ with $\phi(\alpha) = \phi(z)$ $\forall z \in C_2$, then we obtain a new partial Szegö relation in which the test function is a arbitrary continuous function over $C_1 \cup C_3$, but it is constant over the branch C_2. If we sum up these two partial relations, then the general Szegö formula is derived for this specific setting, in which the essential range of the symbol h is Y shaped and compact.

In conclusion, despite the negative answer provided by Proposition 3.5 for satisfying the key assumption of Theorem 3.4, the arguments used for the case where the range of h is contained in a Y shaped compact tells us that the Szegö relation can be extended as long as we have a finite number of branches. The general case cannot be covered in the same fashion. However, the indications collected so far allows one to hope for a positive answer in the general setting, i.e., when considering a cumulative symbol in the Tilli class.

4. Applications to approximated BVPs

Here we treat the case of boundary value problems (BVPs) approximated by finite difference (FD) schemes. The idea is very general and can be applied to other local approximation methods as finite elements or finite volumes on regions of $\mathbb{R}^d$, $d \geqslant 1$. However, for the sake of clarity, we choose a single elliptic one-dimensional BVP for illustrating the general strategy. Let a, b, c, and γ be four given continuous functions on $[0, 1]$ and let us consider the second-order differential equation $-(au')' + bu' + c = \gamma$ on $(0, 1)$, with given Neumann–Dirichlet boundary conditions, i.e., $u(0) = \alpha$, $u'(1) = 0$. We consider the centered FD formula of minimal bandwidth, with precision order two and on the equi-spaced grid $\{x_j\}_{j=0}^{n+1}$, $x_j = jh$, $h = (n + 1)^{-1}$.

The coefficient matrix of the resulting linear system has the form

$$X_n = A'_n(a) + R_n, \quad A'_n(a) = A_n(a) - a_{n-\frac{1}{2}} e_n e_{n-1}^T,$$

$R_n = h^2 \operatorname{diag}_{j=1,\ldots,n}(c_j) + h \operatorname{diag}_{j=1,\ldots,n}(b_j) \quad T_n(\mathrm{i}\sin(t))$, $v_j = v(x_j)$, $\quad v \in \{a, b, c\}$, and where

$$A_n(a) = \begin{bmatrix} a_{\frac{1}{2}} + a_{\frac{3}{2}} & -a_{\frac{3}{2}} & & & & \\ -a_{\frac{3}{2}} & a_{\frac{3}{2}} + a_{\frac{5}{2}} & -a_{\frac{5}{2}} & & & \\ & -a_{\frac{5}{2}} & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -a_{n-\frac{1}{2}} \\ & & & & -a_{n-\frac{1}{2}} & a_{n-\frac{1}{2}} + a_{n+\frac{1}{2}} \end{bmatrix}.$$

From the previous representation and taking into account that $nh < 1$, it easy to check that

$$\|h^2 \operatorname{diag}_{j=1,\ldots,n}(c_j)\| \leqslant h^2\|c\|_\infty, \quad \|h \operatorname{diag}_{j=1,\ldots,n}(b_j)\, T_n(\mathrm{i}\sin(t))\| \leqslant h\|b\|_\infty,$$

and

$$\|A_n'(a) - A_n(a)\| \leqslant \|a\|_\infty,$$

with $\operatorname{rank}(A_n'(a) - A_n(a)) \leqslant 1$, and hence the coefficient matrix X_n equals $A_n(a)$ plus a correction whose trace norm is bounded by $\|a\|_\infty + \|b\|_\infty + h\|c\|_\infty$. Furthermore, as clearly analyzed in [19], the Hermitian (real symmetric indeed) sequence $\{A_n(a)\}$ is distributed as $a(x)(2 - 2\cos(s))$ over $(0, 1) \times (0, \pi)$. As a consequence, by Theorem 2.3, it follows that the non-Hermitian sequence $\{X_n\}$ shares the same distribution function with $\{A_n(a)\}$. In addition, since the trace norm of the correction $X_n - A_n(a)$ is bounded by a pure constant, for every fixed $\epsilon > 0$ it follows that the number of eigenvalues of X_n not belonging to an ϵ-neighborhood of the range of $a(x)(2 - 2\cos(s))$ is bounded by a constant, possibly depending on ϵ, but independent of n (see Theorem 3.5 in [7]).

It is worth noticing that all these derivations go through, with minor modifications, also for higher order differential operators, in higher dimension, and by varying the boundary conditions: as an example, the previous analysis remains the same if the Neumann–Dirichlet boundary conditions are replaced by Dirichlet boundary conditions. As it can be easily argued, the only significant exception can be found when considering somehow artificial singularly perturbed problems in which the perturbation parameter is of the order of h (or of some power of h): in that case the analysis becomes more involved, since the arising matrix structures become significantly non-normal and different tools have to be taken into consideration.

5. Concluding remarks

We have extended a recent perturbation result based on a theorem by Mirsky. More in detail our findings concern the eigenvalue distribution and localization of a generic (non-Hermitian) complex perturbation of a bounded Hermitian sequence of matrices. Some examples of application have been studied, ranging from the product of Toeplitz sequences to the approximation of PDEs with given boundary conditions. As an important example of application, it would be nice to extend the CG convergence analysis of Beckermann and Kuijlaars [1] to this quasi-Hermitian setting, in the case where the Hermitian part is positive definite for every n and the global distribution function of the eigenvalues is positive and bounded. The key point would be the definition of proper assumptions on the outliers and on the extreme eigenvalues in order to mimic, if possible, the same analysis performed in [1].

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